Closing Tue: 12.1, 12.2, 12.3
Closing Thu: 12.4(1), 12.4(2), 12.5(1)

12.4 The Cross Product

We define the <u>cross product</u>, or <u>vector product</u>, for two 3-dimensional vectors, $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$, by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}$$

$$= (a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{j} + (a_1 b_2 - a_2 b_1)\boldsymbol{k}$$

Ex: If $\boldsymbol{a} = \langle 1,2,0 \rangle$ and $\boldsymbol{b} = \langle -1,3,2 \rangle$, then

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 0 \\ -1 & 3 & 2 \end{vmatrix} =$$

$$(\qquad - \qquad)\boldsymbol{i} - (\qquad - \qquad)\boldsymbol{j} + (\qquad - \qquad)\boldsymbol{k}$$

You do $\boldsymbol{a} = \langle 1,3,-1 \rangle$, $\boldsymbol{b} = \langle 2,1,5 \rangle$.
Compute $\boldsymbol{a} \times \boldsymbol{b}$

Most important fact:
The vector $\boldsymbol{v} = \mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**.

Right-hand rule

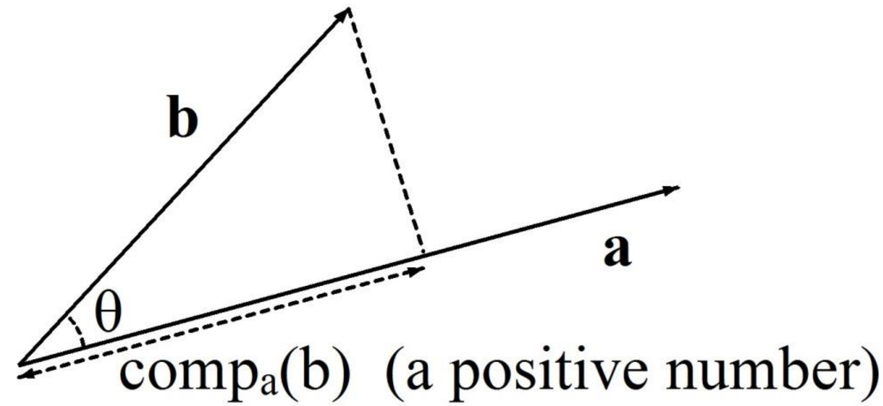
If the fingers of the right-hand curl from **a** to **b**, then the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$*:*

Through some algebra and using the dot product rule, it can be shown tha

$$|\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta)$$

where θ is the smallest angle betwee $\boldsymbol{a}$ and $\boldsymbol{b}$**.** $(0 \leq \theta \leq \pi)$



12.5 Intro to Lines in 3D

To describe 3D lines we use parametric equations. Here is a 2D example:

Ex: Consider the 2D line: $y = 4x + 5$.

(a) Find a vector parallel to the line.
Call it vector **v.**

(b) Find a vector whose head touches the line when drawn from the origin.
Call it vector $\mathbf{r_0}$.

(c) We can reach all other points on the line by walking along $\mathbf{r_0}$, then adding scale multiples of **v**.

This same idea works to describe any line in 2- or 3-dimensions.

The equation for a line in 3D:

$\boldsymbol{v} = \langle a, b, c \rangle$ = parallel to the line.

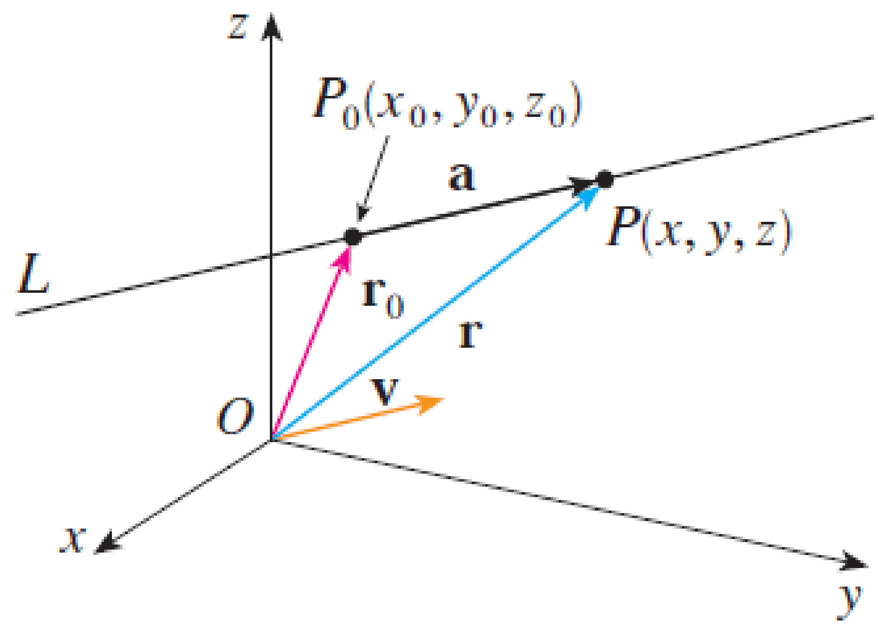
$\boldsymbol{r_0} = \langle x_0, y_0, z_0 \rangle$ = a position vector

then all other points, (x, y, z), satisfy

$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \mathrm{t}\langle a, b, c \rangle$,

for some number *t*.

The above form ($\boldsymbol{r} = \boldsymbol{r_0} + \mathrm{t}\,\boldsymbol{v}$) is called the *vector form* of the line.



We also write this in *parametric form* as:

$$x = x_0 + at,$$
$$y = y_0 + bt,$$
$$z = z_0 + ct.$$

or in *symmetric form*:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Basic Example – Given Two Points:
Find parametric equations of the line thru the points P(1,0,2) and Q(-1, 2, 1).

General Line Facts

1. Two lines are **parallel** if their direction vectors are parallel.

2. Two lines **intersect** if they have an (x,y,z) point in common (use different different parameters!)

Note: The *acute angle of intersection* would be the acute angle between the direction vectors.

3. Two lines are **skew** if they don't intersect and aren't parallel.